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Pivoting in an Outcome Polyhedron

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Abstract. In many types of linear, convex and nonconvex optimization problems over polyhedra, a global optimal solution can be found by searching the extreme points of the outcome polyhedron Y instead of the extreme points of the decision set polyhedron Z. Since the dimension of Y is often significantly smaller than the dimension of Z, and since the structure of Y is often much simpler than the structure of Z, such an approach has the potential to often yield significant computational savings. This article seeks to motivate these potential savings through both general theory and concrete examples. The article then develops two new procedures. The first procedure is linear-programming based and finds an initial extreme point of an outcome polyhedron Y. The second procedure provides a mechanism for moving from a given extreme point y of Y along any chosen edge of Y emanating from y until a neighboring extreme point to y is reached. As a by-product of the second procedure, as in the pivoting process of the simplex method, a complete algebraic description of the chosen edge can also be easily obtained.

Key words: Outcome polyhedron; Linear programming; Pivoting; Nonlinear programming; Global optimization; Extreme point mathematical programming; Neighborhood problem

1. Introduction

Many types of single and multiple objective linear, convex and global optimization problems (P) can be written

$$\min g(z), \text{ s.t. } z \in Z ,$$

where Z is a nonempty, compact polyhedral feasible decision set in $\Re^n$, and g is either a single-valued or multi-valued mapping defined on a suitable subset of $\Re^n$. For a significant number of these problems, by defining a suitable $p \times n$ matrix D, an optimal solution in Z can be found by minimizing an appropriate mapping $\bar{g}(y)$ over the outcome polyhedron (or outcome set)

$$Y = \{y \in \Re^p \mid y = Dz \text{ for some } z \in Z\} , \tag{1}$$

rather than by examining or searching Z. Notice that this is equivalent to solving the problem

$$\begin{aligned} &\min \bar{g}(y) , \\ &\text{s.t.} \quad y = Dz , \\ &z \in Z , \end{aligned}$$

although we focus on the formulation where we minimize $\bar{g}(y)$ over Y, where Y is defined by (1), to emphasize the role of the outcome polyhedron Y.

To globally solve problem (P), significant computational savings can potentially be achieved by minimizing $\bar{g}(y)$ over Y instead of by directly examining or searching Z. This is in part because, as we shall see, p is typically much smaller than n, so that the outcome polyhedron Y will typically have a much smaller dimension than Z. It is also because, as we shall see, Y can be expected to have a far simpler structure than Z. Furthermore, in many cases, the global optimal value of $\bar{g}(y)$ over the outcome polyhedron Y is guaranteed to exist at an extreme point of Y. As we shall see, Y can be expected to have far fewer faces, including fewer extreme points, than Z. As a result, in these cases, the opportunity arises to employ simplex method-type pivoting among the extreme points of Y to help to globally optimize $\bar{g}(y)$ over Y. For these cases, such an approach can potentially yield especially significant computational savings.

This article has three purposes. First, it seeks, through both general theory and concrete examples, to motivate the potential benefits to be accrued by developing procedures for finding an extreme point and pivoting in a simplex-method manner among the extreme points of the outcome polyhedron Y. Second, as a first step for such procedures, the article develops and validates a linear programming-based method for finding an initial extreme point of the outcome polyhedron Y. Third, the article develops mechanics suitable for moving from a given extreme point y of Y along any chosen edge of Y emanating from y until the extreme point neighbor to y along this edge is reached. As a by-product of this pivoting process, as in the pivoting process of the simplex method, a complete algebraic description of the chosen edge can also be easily obtained. This part of the article was motivated, in part, by some related work of Dauer and Liu [9].

The organization of this article is as follows. In Section 2, to motivate the research to be presented, we demonstrate via theoretical arguments and concrete examples the potential benefits to be obtained by developing procedures for pivoting among the extreme points of an outcome polyhedron. In Section 3 we present some theoretical results of potential use for finding an initial extreme point of an outcome polyhedron. Based upon some of these results, we then give and validate a finite, linear programming-based procedure for finding an initial extreme point of the outcome polyhedron Y. In Section 4, after giving some necessary preliminary results, a procedure is presented that, for the first time, solves the neighborhood problem [13, 14, 18] for the outcome polyhedron Y. The mechanics for pivoting in a simplex method-like manner among the neighboring extreme points of Y follow immediately from this procedure. To illustrate the initialization and pivoting procedures for Y, they are applied to small examples in Sections 3 and 5, respectively.

2. Motivation

Throughout this article, Z will denote a nonempty, compact polyhedron in $\Re^n$, and

D will represent a $p \times n$ matrix of real numbers. In addition, let the *outcome polyhedron* (or *outcome set*) Y be defined throughout by

$$Y = \{y \in \Re^p \mid y = Dz \text{ for some } z \in Z\}.$$

For economy of notation, Y is also denoted $Y = D[Z]$. It can be shown that Y is a nonempty, compact polyhedron in $\Re^p$; see, for example, Rockafellar [22, p. 174].

The reason for calling Y an outcome polyhedron is as follows. In many types of single objective and multiple objective optimization problems (P), both convex and nonconvex, the feasible decision set can be represented by linear constraints that define a nonempty, compact polyhedron $Z \in \Re^n$. For a significant number of these problems, if D is defined appropriately, a global optimal solution $z^* \in Z$ and the global optimal objective function value v for problem (P) can be found by globally minimizing a suitable function or mapping $\bar{g}$ of $y \in \Re^p$ over the set $Y = D[Z]$, rather than by minimizing the original objective function or mapping over Z. In this sense, Y represents the feasible outcomes of the decisions $z \in Z$ under the mapping D, and Y is therefore called an outcome polyhedron. In addition, we will refer to the vector or vectors found by globally minimizing $\bar{g}$ over Y as global *optimal outcomes*.

The importance of the outcome polyhedron Y is two-fold. First, Y can be expected to have a significantly smaller dimension and a significantly simpler structure than Z. In particular, the value of p in the $p \times n$ matrix D in optimization applications is frequently significantly smaller than the number of decision variables n. Since, by [5], the dimension of Y cannot exceed the rank of D, this implies that the dimension of Y in applications is never more than p and, hence, can be expected to be significantly smaller than the dimension of Z. Furthermore, from [4], since p is typically much smaller than n, Y can be expected to have far fewer extreme points and far fewer faces than Z. In addition, from [4], since p is typically much smaller than n, the dimensions of the faces of Y can be expected to be far smaller than those of Z, and large numbers of extreme points of Z can be expected to be mapped by D either into a single extreme point of Y or into non-extreme points of Y.

Second, it is often the case that a global optimal outcome $y^* \in Y$ is guaranteed to exist at an extreme point of Y. When this is the case, the opportunity becomes available to solve for a global optimal outcome $y^* \in Y$ by adapting to the outcome set problem any of a large number of domain-based algorithms that rely either partly or completely upon simplex method-like pivoting.

Taken together, these two properties of the outcome polyhedron Y imply that when Z represents the feasible decision set of an optimization problem (P), significant computational savings can be expected to accrue when problem (P) is solvable by minimizing a suitable function or mapping $\bar{g}(y)$ over Y, especially when it can be shown that a global optimal outcome exists at an extreme point of Y. In many cases, the latter property can be shown to hold. For instance, from [5] and [7], this property holds for certain linear programming problems, in multiple objective linear programming, in bilinear programming, in problems of optimization over efficient sets, and in linear multiplicative programming.

In order to demonstrate that, in practice, having available a mechanism for pivoting among the extreme points of Y can potentially yield very efficient methods computationally for globally solving problem (P), let us consider in some detail two of the problem classes mentioned above. For additional examples, see the working papers [5] and [7]. For any convex set W, let W_{ex} represent the set of all extreme points of W.

EXAMPLE 2.1. Bilinear Programming. A typical *bilinear programming problem* has the form

$$v_1 = \min \langle c, x\rangle + x^T \bar{D} z + \langle d, z\rangle\,, \qquad \text{(PZ1)}$$
$$\text{s.t.}$$
$$x \in X, z \in Z\,,$$

where $X \subseteq \Re^{\bar{p}}$ is a nonempty, compact polyhedron, $\bar{D}$ is a $\bar{p} \times n$ matrix of real numbers, $c \in \Re^{\bar{p}}$, $d \in \Re^n$ and, as before, Z is a nonempty, compact polyhedron in $\Re^n$. The bilinear programming problem is one of the oldest and most frequently encountered global optimization problems in mathematical programming. It was first studied in the 1960s. At that time, proposed solution methods were either locally convergent or completely enumerative. Subsequently, global solution procedures using extreme point ranking, relaxation methods, cutting planes, or branch and bound approaches were proposed. Applications of bilinear programming have been made to bimatrix games, to certain assignment problems, to multicommodity network flow problems, and to other areas of production and planning. For more details concerning bilinear programming, see Horst and Tuy [15], Konno et al. [17], Quesada and Grossmann [21] and references therein.

It is not difficult to show that bilinear program (PZ1) has a global optimal solution (x^*, z^*) such that $x^* \in X_{\mathrm{ex}}$ and $z^* \in Z_{\mathrm{ex}}$ [17, Proposition 7.4]. Indeed, certain solution algorithms for bilinear programming take advantage of this property. However, these algorithms also may work in $\Re^{\bar{p}+n}$ to globally solve problem (PZ1), where $(\bar{p} + n)$ may be relatively large.

Let $p = \bar{p} + 1$, and set $Y = D[Z]$, where D is the $p \times n$ matrix with row i equal to row i of $\bar{D}$ for each $i = 1, 2, \ldots, \bar{p}$, and with row p equal to d^T. Let a typical element y of Y be represented by $y^T = (\bar{y}^T, y_p)$, where $\bar{y}_i^T = y_i$, $i = 1, 2, \ldots, \bar{p}$. Then, following an approach similar to that of Konno et al. [17, pp. 343–344], we may write v_1 as

$$v_1 = \min \langle c, x\rangle + \langle \bar{y}, x\rangle + y_p\,,$$
$$\text{s.t.}$$
$$x \in X, y \in Y\,,$$
$$= \min_{y \in Y} [y_p + \min_{x \in X} (\langle \bar{y}, x\rangle + \langle c, x\rangle)],$$

$$= \min_{y \in Y} \bar{g}(y) , \tag{2}$$

where $\bar{g} : Y \rightarrow \Re$ is defined for each $y \in Y$ by

$$\bar{g}(y) = y_p + \min_{x \in X}(\langle \bar{y}, x \rangle + \langle c, x \rangle) . \tag{3}$$

Since $\bar{g} : Y \rightarrow \Re$ is a concave function, the global minimum of $\bar{g}$ over Y is attained at an extreme point of Y [19]. Furthermore, from (3) and Benson [4, p. 237], since $Y = D[Z]$ and Z is compact, for each global optimal solution $y^* \in Y_{\text{ex}}$ for (2), there exists a point $x^* \in X_{\text{ex}}$ and a point $z^* \in Z_{\text{ex}}$ such that x^* solves the minimization problem in (3), $y^* = Dz^*$ and (x^*, z^*) is a global optimal solution for problem (PZ1).

The arguments above demonstrate that by solving the concave minimization, outcome set-based problem (PY1) given by (2) for an extreme point global optimal solution, the decision set-based problem (PZ1) is also globally solved. Since $Y \subseteq \Re^p$, where $p = (\bar{p} + 1)$ is typically smaller than $\bar{p} + n$, from our earlier discussions in this section, the reduction in dimensionality obtained by using this outcome set approach can potentially be quite significant computationally, especially because problem (PZ1) is a global optimization problem.

To globally solve problem (PY1), one could potentially adapt as needed the approaches of a number of concave minimization algorithms designed for solving decision set-based problems. Included among these are several algorithms that rely significantly upon procedures for pivoting among the neighboring extreme points of a polyhedron [2, 3, 15]. To adapt algorithms such as these to globally solving problem (PY1), mechanisms for pivoting among the extreme points of the outcome polyhedron Y will be needed. As we have seen, such an adaptation has the potential to yield significant computational savings.

EXAMPLE 2.2. Linear Multiplicative Programming. The *linear multiplicative programming* problem may be written

$$v_2 = \min \prod_{i=1}^{p} \langle D_i, z \rangle, \text{ s.t. } z \in Z , \tag{PZ2}$$

where, for each $i = 1, 2, \ldots, p$, D_i represents row i of D and for each $z \in Z$, $\langle D_i, z \rangle > 0$. Notice in problem (PZ2) that the minimum v_2 is achieved. It is well-known that problem (PZ2) generally has many local optima that are not global optima. In recent years, a resurgence of interest in problem (PZ2) has occurred [6, 11, 16, 17, 23, 24, 26, 27]. This is due, in part, to the variety of applications of the problem. It is also because global optimization codes and rapid advances in high-speed computing are now allowing for the global solution of problem (PZ2) for the first time [17].

It can be shown [6, p. 493], that $g : Z \rightarrow \Re$ defined for each $z \in Z$ by

$$g(z) = \prod_{i=1}^{p} \langle D_i, z \rangle$$

is a quasiconcave function. As a result, problem (PZ2) has a global optimal solution that belongs to Z_{ex} [19].

Since $Y = D[Z]$, we have

$$\begin{aligned} v_2 &= \min \prod_{i=1}^{p} \langle D_i, z \rangle, \text{ s.t. } z \in Z \\ &= \min \prod_{i=1}^{p} y_i, \text{ s.t. } y \in Y . \end{aligned} \tag{4}$$

Because the function $\bar{g} : Y \rightarrow \Re$ defined for each $y \in Y$ by

$$\bar{g}(y) = \prod_{i=1}^{p} y_i$$

is quasiconcave [6, p. 493], the global optimization problem (PY2) in (4) has a global optimal solution $y^* \in Y_{ex}$ [15]. Furthermore, from (4) and Benson [4, p. 237], since Z is compact, for each global optimal solution $y^* \in Y_{ex}$ for problem (PY2), there will exist a point $z^* \in Z_{ex}$ such that $g(z^*) = \bar{g}(y^*)$. It follows that to find a global extreme point optimal solution for problem (PZ2), one may instead solve the optimization problem (PY2) for a global extreme point optimal solution.

Since p is typically considerably smaller than n in applications of the linear multiplicative programming problem (PZ2), the potential benefits of globally solving problem (PY2) instead of problem (PZ2) are significant, especially because problem (PZ2) is a global optimization problem. To globally solve problem (PY2), one may modify as needed the mechanics of any of a number of concave minimization algorithms that have been designed for problems such as problem (PZ2) [2, 3, 15]. Several of these algorithms involve, in part, pivoting from extreme point to neighboring extreme point in Z. Therefore, if these types of algorithms are modified and applied to solving problem (PY2), mechanisms for pivoting among the extreme points of the outcome polyhedron Y will be needed. As we have seen, using such mechanisms can potentially yield significant computational savings.

3. Finding an initial extreme point of *Y*

Since Y is a nonempty, compact polyhedron, Y_{ex} is a nonempty set [22, p. 167]. Furthermore, because Z is compact and $Y = D[Z]$, for each element y of Y_{ex}, there exists at least one element z of Z_{ex} such that $y = Dz$ [4, p. 237]. Unfortunately, however, not every element z of Z_{ex} satisfies the condition that $y = Dz \in Y_{ex}$. In fact, impressively-large numbers of elements of Z_{ex} can be mapped by D into non-extreme points of Y (for details and examples, see Benson [4]). In this section, we develop some necessary and sufficient conditions for a point $z \in Z_{ex}$ to satisfy $y \in Y_{ex}$, where $y = Dz$. Based upon some of these results, we then give and validate a finite, linear programming-based procedure that is guaranteed to find a point $y \in Y_{\mathrm{ex}}$. For brevity, the reader will be referred to [5] for proofs of several of the results given in this section.

Throughout the remainder of the article we will assume, without loss of generality, for some $m \times n$ matrix A with rank m, where $m \leqslant n$, and for some vector $b \in \Re^m$, that Z can be represented by

$$Z = \{z \in \Re^n \mid Az = b, z \geqslant 0\} .$$

Recall from linear programming theory and the simplex method of linear programming that if $z \in Z_{\mathrm{ex}}$, then we may choose an invertible $m \times m$ *basis* matrix B for z consisting of m columns of A such that

$$z_B = B^{-1}b - B^{-1}Nz_N ,$$

where N is the $(n-m) \times m$ matrix consisting of the columns of A that do not belong to B, $z_B \in \Re^m$ is the vector of *basic variables* corresponding to B, and $z_N = 0 \in \Re^{n-m}$ is the vector of *nonbasic variables* [12, pp. 19–21]. If, in addition, $y = Dz$, then it follows that

$$y = D_B B^{-1}b - Rz_N ,$$

where D_B is the $p \times m$ matrix of the columns of D corresponding to B, and R is the $p \times (n-m)$ *reduced cost* matrix given by

$$R = D_N - D_B B^{-1}N ,$$

where D_N is the $p \times (n-m)$ matrix consisting of the columns of D that do not belong to D_B. Let r_j, $j = 1, 2, \ldots, n-m$, denote the columns of R, and let $J = \{j \in \{1, 2, \ldots, n-m\} \mid r_j \neq 0\}$. Notice that if $R = 0$, then Y consists of the single point y, so that $y \in Y_{\mathrm{ex}}$. Therefore, we can assume henceforth that $R \neq 0$, so that $J \neq \emptyset$. Recall that if $z \in Z_{\mathrm{ex}}$, and if B is a basis matrix for z, then B is said to be *nondegenerate* when $z_B > 0$.

THEOREM 3.1. *Assume that $z \in Z_{\mathrm{ex}}$ and that $y = Dz$. Let B denote a basis matrix for z, and consider the linear system*

$$\sum_{j \in J} \alpha_j r_j = 0 , \tag{5}$$

$$\alpha_j \geqslant 0 , \quad j \in J . \tag{6}$$

(*a*) *If the unique solution to the linear system* (5)–(6) *is* $\alpha_j = 0$, $j \in J$, *then* $y \in Y_{\mathrm{ex}}$.

(*b*) If $y \in Y_{\mathrm{ex}}$ *and B is nondegenerate, then the unique solution to the linear system* (5)–(6) *is* $\alpha_j = 0$, $j \in J$.

Proof. See [5]. □

REMARK 3.1. When the unique solution to the linear system (5)–(6) is $\alpha_j = 0$, $j \in J$, then the set $\{r_j \mid j \in J\}$ is called a *positively independent set* of vectors [10].

As a result of Theorem 3.1, we obtain the following corollaries.

COROLLARY 3.1. *Assume that* $z \in Z_{\mathrm{ex}}$ *and that* $y = Dz$. *Let B denote a basis matrix for z, and consider the linear inequality system*

$$\langle r_j, w \rangle \geqslant 1 , \quad j \in J . \tag{7}$$

(a) *If* (7) *has at least one solution, then* $y \in Y_{\mathrm{ex}}$.
(b) *If* $y \in Y_{\mathrm{ex}}$ *and B is nondegenerate, then* (7) *has at least one solution.*
Proof. See [5]. □

From the simplex method, when $z \in Z_{\mathrm{ex}}$, we know that a sufficient condition for (7) to have a solution is that z uniquely minimize $w^T D\bar{z}$ over $\bar{z} \in Z$ for some $w \in \Re^p$. This yields the following result.

COROLLARY 3.2. *Assume that* $z \in Z_{\mathrm{ex}}$ *and that* $y = Dz$. *If z is the unique optimal solution to the linear programming problem*

$$\min w^T D\bar{z}, \text{ s.t. } \bar{z} \in Z \tag{P_w}$$

for some $w \in \Re^p$, *then* $y \in Y_{\mathrm{ex}}$.

Notice that if $z \in Z_{\mathrm{ex}}$ and $y = Dz \in Y_{\mathrm{ex}}$, then it is not necessary for z to be a unique optimal solution to problem (P_w) for some $w \in \Re^p$. This is demonstrated by the following example.

EXAMPLE 3.1. Assume that $n \geqslant 3$, let

$$Z = \{z \in \Re^{2n} \mid z_j + z_{j+n} = 1,\ j = 1, 2, \ldots, n,\ z \geqslant 0\} ,$$

and let D be the $2 \times 2n$ matrix whose first two columns form the 2×2 identity matrix and whose remaining columns are all zeroes. Then $Y = D[Z]$ is a unit square in $\Re^2$, and $y^T = (1, 1)$ satisfies $y \in Y_{\mathrm{ex}}$. Notice that if we define $z \in \Re^{2n}$ by

$$z_j = \begin{cases} 1 & \text{if } 1 \leqslant j \leqslant n , \\ 0 & \text{if } (n+1) \leqslant j \leqslant 2n , \end{cases}$$

then $z \in Z_{\mathrm{ex}}$ and $y = Dz$. However, it is easy to see that for each $w \in \Re^2$ such that z is an optimal solution to the linear program (P_w), problem (P_w) has multiple optimal solutions.

From linear programming theory, we know that for any fixed $w \in \Re^p$, if z is the unique optimal solution to problem (P_w), then $z \in Z_{\mathrm{ex}}$. Combined with Corollary 3.2, this implies that if, for some $w \in \Re^p$, the linear program (P_w) in Corollary 3.2 has a unique optimal solution z, then $y = Dz$ is an extreme point of Y. This raises the question of whether or not an element of Y_{ex} can be found when for some $w \in \Re^p$, the linear program (P_w) has multiple optimal solutions. The next result answers this question, at least in theory.

THEOREM 3.2. *Consider the linear programming problem* (P_w) *defined in Corollary 3.2. Suppose that for some* $w \in \Re^p$, *problem* (P_w) *has multiple optimal solutions. Then there exists an optimal solution* $z \in Z_{ex}$ to problem (P_w) *such that* $y = Dz \in Y_{ex}$.

Proof. See [5].

REMARK 3.2. Since Z is bounded, whenever the linear program (P_w) in Corollary 3.2 has multiple optimal solutions for some $w \in \Re^p$, it will have multiple optimal solutions that are extreme points of Z [20, pp. 131, 139]. Although methods exist for iteratively finding all extreme point optimal solutions for a linear program, these methods are relatively inefficient computationally [25, Ch. 4]. To use Theorem 3.2 in practice to help to find an element of Y_{ex} would require not only using one of these methods to iteratively find alternate optimal solutions $z \in Z_{ex}$ for problem (P_w), but also testing each such point z found to see if $y = Dz \in Y_{ex}$. Thus, the implementation of Theorem 3.2 in practice may be rather cumbersome, at best.

The following result will be important in helping to develop the procedure for finding an element of Y_{ex}.

THEOREM 3.3. *For each* $i = 1, 2, \ldots, p$, *let* D_i *denote row* i *of* D. *Let* v_1 *denote the optimal value for the linear program*

$$\min\langle D_1, z\rangle \text{ s.t. } z \in Z , \tag{LPD$_1$}$$

and, for each $i = 2, 3, \ldots, \hat{\imath}$, *let* v_i *denote the optimal value for the linear program*

$$\begin{aligned} &\min\langle D_i, z\rangle , && \text{(LPD}_i\text{)} \\ &\text{s.t.} \\ &\langle D_k, z\rangle = v_k ,\ k = 1, 2, \ldots, i-1 , \\ &z \in Z , \end{aligned}$$

where $2 \leqslant \hat{\imath} \leqslant p$. *Suppose that* $i \in \{1, 2, \ldots, \hat{\imath}\}$. *If* z^i *is the unique optimal solution to problem* (LPD_i), *then* $z^i \in Z_{ex}$ *and* $Dz^i \stackrel{\triangle}{=} y^i \in Y_{ex}$.

Proof. Suppose that $y^i = \alpha\bar{y} + (1-\alpha)\bar{\bar{y}}$ for some $\bar{y}, \bar{\bar{y}} \in Y$ and $\alpha \in \Re$ such that $0 < \alpha < 1$, where $1 \leqslant i \leqslant \hat{\imath}$. Since $\bar{y}, \bar{\bar{y}} \in Y$, $D\bar{z} = \bar{y}$ and $D\bar{\bar{z}} = \bar{\bar{y}}$ for some $\bar{z}, \bar{\bar{z}} \in Z$. Therefore, by definition of y^i,

$$Dz^i = \alpha D\bar{z} + (1-\alpha)D\bar{\bar{z}} .$$

From this equation, since z^i is an optimal solution to problem (LPD_i), we obtain that for each $k = 1, 2, \ldots, i$,

$$v_k = \langle D_k, z^i \rangle = \alpha \langle D_k, \bar{z} \rangle + (1-\alpha)\langle D_k, \bar{\bar{z}} \rangle . \quad (8)$$

With $k = 1$, since $\bar{z}, \bar{\bar{z}} \in Z$ and $0 < \alpha < 1$, (8) implies that $\bar{z}$ and $\bar{\bar{z}}$ are optimal solutions to problem (LPD_1). As a result, $\bar{z}$ and $\bar{\bar{z}}$ are feasible solutions to problem (LPD_2). By setting $k = 2$ in (8), since $0 < \alpha < 1$, this implies together with (8) that $\bar{z}$ and $\bar{\bar{z}}$ are optimal solutions to problem (LPD_2). As a result, $\bar{z}$ and $\bar{\bar{z}}$ are feasible solutions to problem (LPD_3). With $k = 3$ in (8), this implies in a similar manner that $\bar{z}$ and $\bar{\bar{z}}$ are optimal solutions to problem (LPD_3). By continuing in this fashion, we see that $\bar{z}$ and $\bar{\bar{z}}$ are optimal solutions to problem (LPD_i). Since z^i is the unique optimal solution to problem (LPD_i), it follows that $\bar{z} = \bar{\bar{z}} = z^i$. Therefore, $\bar{y} = \bar{\bar{y}} = y^i$, so that, by definition, $y^i \in Y_{\mathrm{ex}}$.

Since z^i is an optimal solution to linear program (LPD_i), $z^i \in Z$ and

$$\langle D_k, z^i \rangle = v_k, \ \ k = 1, 2, \ldots, i-1 .$$

It follows that z^i is an optimal solution to linear program (LPD_k) for each $k = 1, 2, \ldots, i$. For each $k \in \{2, 3, \ldots, i\}$, from [20, pp. 139–141], the optimal solution set of linear program (LPD_k) is a (polyhedral) face of the polyhedral feasible region of problem (LPD_{k-1}), and the optimal solution set of linear program (LPD_1) is a (polyhedral) face of Z. Therefore, from Rockafellar [22, p. 163], the optimal solution set Z_i^* of problem (LPD_i) is a face of Z. Since z^i is the unique optimal solution to problem (LPD_i), this implies that z^i is an extreme point of Z. □

Assume without loss of generality that rank $D = q$, where $1 \leqslant q \leqslant p$ and that $\{D_i \mid i = 1, 2, \ldots, q\}$ is a set of any q linearly independent rows of D. We may now present the following simple procedure, which is guaranteed to find an element of Y_{ex}. It uses the linear programming problems (LPD_i), $i = 1, 2, \ldots, q$, each of which is defined in Theorem 3.3.

Step 1. Find any point $z^1 \in Z_{\mathrm{ex}}$ that is an optimal solution to the linear program (LPD_1). If z^1 is the unique optimal solution to problem (LPD_1), then STOP: $y^1 \triangleq Dz^1 \in Y_{\mathrm{ex}}$. Otherwise, let v_1 denote the optimal value of problem (LPD_1), set $i = 2$, and go to Step i.

Step i ($i = 2, 3, \ldots, q$). Find an optimal solution z^i to the linear program (LPD_i). If z^i is the unique optimal solution to problem (LPD_i), or if $i = q$, then STOP: $y^i \triangleq Dz^i \in Y_{\mathrm{ex}}$. Otherwise, let v_i denote the optimal value of problem (LPD_i), set $i = i + 1$, and go to Step i.

THEOREM 3.4. *The procedure above terminates in Step i with a point* $y^i \in Y_{\mathrm{ex}}$, *where i is some element of* $\{1, 2, \ldots, q\}$.

Proof. If the procedure stops in Step 1, then $y^1 = Dz^1 \in Y_{\mathrm{ex}}$ by Corollary 3.2. If the procedure stops in Step i for some i such that $2 \leqslant i \leqslant q-1$, then, by Theorem 3.3, $y^i = Dz^i \in Y_{\mathrm{ex}}$. If the procedure stops in Step q and z^q is the unique optimal solution to problem (LPD_q), then $y^q = Dz^q \in Y_{\mathrm{ex}}$ by Theorem 3.3.

Now suppose that the procedure stops in Step q, but z^q is not the unique optimal solution for problem (LPD$_q$). Then, using logic similar to the logic used in the proof of Theorem 3.3, it follows that $y^q \triangleq Dz^q$ is an optimal solution to the linear program

$$\begin{aligned} &\min\, y_q\,, && (\mathrm{LPY}_q)\\ &\text{s.t.}\\ &y_k = v_k\,,\ k = 1, 2, \ldots, q-1\,,\\ &y \in Y\,. \end{aligned}$$

Suppose for some $\bar{y}$, $\bar{\bar{y}} \in Y$ and $\alpha \in \Re$ such that $0 < \alpha < 1$ that $y^q = \alpha\bar{y} + (1 - \alpha)\bar{\bar{y}}$. By using logic similar to that used in the proof of Theorem 3.3, it follows that $\bar{y}$ and $\bar{\bar{y}}$ are optimal solutions to problem (LPY$_q$), and that for each $k = 1, 2, \ldots, q$,

$$y_k^q = \bar{y}_k = \bar{\bar{y}}_k\,. \tag{9}$$

Since $\bar{y}$, $\bar{\bar{y}} \in Y$, we may choose $\bar{z}$, $\bar{\bar{z}} \in Z$ such that $\bar{y} = D\bar{z}$ and $\bar{\bar{y}} = D\bar{\bar{z}}$. From (9), this implies that for each $k = 1, 2, \ldots, q$,

$$\langle D_k, z^q \rangle = \langle D_k, \bar{z} \rangle = \langle D_k, \bar{\bar{z}} \rangle\,. \tag{10}$$

Because rank $D = q$ and $\{D_k \mid k = 1, 2, \ldots, q\}$ is a linearly independent set, from (10) it is easy to show that for each $k \not\in \{1, 2, \ldots, q\}$,

$$\langle D_k, z^q \rangle = \langle D_k, \bar{z} \rangle = \langle D_k, \bar{\bar{z}} \rangle\,.$$

Combined with (10), this implies that $Dz^q = D\bar{z} = D\bar{\bar{z}}$. Therefore, $y^q = \bar{y} = \bar{\bar{y}}$, so that $y^q \in Y_{\mathrm{ex}}$.

Since q is a finite number, it is clear that the procedure terminates after some finite number $i \leqslant q$ of steps have been executed. Combined with the arguments above, this completes the proof. □

EXAMPLE 3.2. Let I_{10} denote the 10×10 identity matrix. Let

$$Z = \{z \in \Re^{20} \mid Az = b,\ z \geqslant 0\}\,,$$

where A is the 10×20 matrix

$$A = [I_{10} \,\vdots\, I_{10}]$$

and $b \in \Re^{10}$ is the vector whose entries are each equal to 1.0. Suppose that $Y = D[Z]$, where D is the 2×20 matrix of rank 2 with columns 1 and 2 equal to $e \in \Re^2$, where e denotes the vector with each entry equal to 1.0, with columns 3 through 10 equal to e^2, where

$$e^2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\,,$$

and with remaining columns 0. It is easy to see that if we view z_j, $j = 11, 12, \ldots, 20$, as slack variables, then Z is a hypercube in $\Re^{10}$ with 1024 extreme points, and $Y = D[Z]$ is a two-dimensional (compact) polyhedron in $\Re^2$.

If we apply the procedure for finding an initial element of Y_{ex} to this example, then, in Step 1, we discover that linear program (LPD_1) has multiple optimal solutions and optimal value $v_1 = 0$. We therefore proceed to Step 2, where we solve linear program (LPD_2). We find that this linear program has a unique optimal solution $z^2 \in \Re^{20}$ given by

$$z_i^2 = \begin{cases} 0\,, & \text{if } 1 \leqslant i \leqslant 10\,, \\ 1\,, & \text{if } 11 \leqslant i \leqslant 20\,. \end{cases}$$

The procedure then stops, returning $y^2 = Dz^2 = [0, 0]^T \in Y_{\mathrm{ex}}$.

REMARK 3.3. In [9, Remark 4.1], Dauer and Liu informally suggest an idea for finding an element of Y_{ex}. However, this idea will not always succeed. For instance, if we apply their idea to Example 4.1 in [9] by choosing the initial extreme point $\hat{z}$ of Z to be

$$\hat{z}^T = [1, 1, 1, 61, 55, 55, 0, 0, 0, 7, 7]\,,$$

then the reduced cost matrix R is given by

$$R = \left(\frac{1}{486}\right) \begin{bmatrix} 71 & -10 & -1 \\ -10 & 71 & -1 \end{bmatrix}.$$

According to Dauer and Liu's idea, since no column of R is a negative multiple of some other column of R, $\hat{y}^T \overset{\triangle}{=} D\hat{z} = (10/9, 10/9)$ should be an extreme point of $Y = D[Z]$. However, from [9], it is easy to see that this is false, since $\hat{y}$ lies in the interior of Y. On the other hand, the procedure given above is fail-safe and, in this case, it is easy to see that it finds the extreme point $y^T = (0.111111, 8.111111)$ of Y.

4. Pivoting among the extreme points of *Y*

Assume that we are given an extreme point y of Y. Then, since Z is compact, there exists a point $z \in Z_{\mathrm{ex}}$ such that $y = Dz$ [4]. In fact, if necessary, we can always find such a point z. For instance, if y is generated by the initialization procedure given in Section 3, and that procedure terminates in Step i, where $i < q$, then from the procedure and Theorem 3.3, it follows that the point z^i generated by the procedure satisfies $z^i \in Z_{\mathrm{ex}}$ and $y = Dz^i$. If the procedure given in Section 3 terminates in Step q, then as we shall see later, by solving a single linear program, a point $z \in Z_{\mathrm{ex}}$ such that $y = Dz$ can be found. Therefore, we may assume that we can find such a point z, if necessary.

In [9], Dauer and Liu suggested that to derive mechanics for pivoting in a simplex method-like manner in Y from $y \in Y_{\mathrm{ex}}$ to any adjacent extreme point to y in Y, one should first solve the neighborhood problem for Y. The *neighborhood problem* [13, 14, 18] in Y is the problem of identifying all of the edges of Y emanating from y and the extreme point neighbors to y in Y that lie at the endpoints of these edges. Given these mechanics, we will be able, starting from y, to repeatedly pivot from extreme point to selected neighboring extreme point in Y. We shall adopt this general approach.

The neighborhood problem has been solved for cases where a linear system of equations and inequalities that represents the polyhedron of interest is available. Since no such system is readily available for Y, we must develop new mechanics in order to solve the neighborhood problem in Y.

To solve the neighborhood problem for $y \in Y_{\text{ex}}$, we will need to address several questions. For instance, will it be necessary to find a point $z \in Z_{\text{ex}}$ such that $Dz = y$ in order to solve the neighborhood problem for y? From the discussion above, we know that this can be done, if necessary. However, there may be multiple points z in Z_{ex} such that $Dz = y$ [4]. In such cases, which point or points of this type, if any, will be needed? How should we proceed if the point or points needed (or found) are nondegenerate? How should we proceed if one or more of these points is degenerate?

To help lay some groundwork for answering these questions, we will look further into some notions studied in [9] and in [28]. This will involve a discussion of cones and some related concepts. Towards this end, recall that a *cone* is a subset W of $\Re^n$ such that $\lambda w \in W$ whenever $w \in W$ and $\lambda \geqslant 0$. A cone W is said to be *pointed* when $W \cap (-W) = \{0\}$. If s^j, $j = 1, 2, \dots, p$, are vectors in $\Re^n$, then the *cone* $W(S)$ *generated by* $S = \{s^1, s^2, \dots, s^p\}$ is defined by

$$W(S) = \left\{ \sum_{i=1}^{p} \lambda_i s^i \mid \lambda_i \geqslant 0 , \quad i = 1, 2, \dots, p \right\} .$$

If $S = \{s^1, s^2, \dots, s^p\}$ is a subset of $\Re^n$, then a subset T of S is called a *frame* for $W(S)$ when $W(T) = W(S)$ and, for each $i \in \{1, 2, \dots, p\}$ such that $s^i \in T$, $W(T \setminus \{s^i\}) \neq W(T)$.

REMARK 4.1. A set $S = \{s^1, s^2, \dots, s^p\}$ may contain more than one frame, and the cardinalities of these frames may differ [28]. However, from [7], when $W(S)$ is a pointed cone, then any frame T for $W(S)$ is of the form

$$T = \{\theta_i u^i \mid i = 1, 2, \dots, t\}$$

for some $\theta_i > 0$, $i = 1, 2, \dots, t$, where u^i, $i = 1, 2, \dots, t$, are the extreme directions of $W(S)$. Given S, Wets and Witzgall [28] give a 'primal' algorithm for finding a frame for $W(S)$, under a certain nondegeneracy assumption. As noted by Wets and Witzgall, it is easy to modify their primal algorithm or to create a 'dual' algorithm that requires no special nondegeneracy assumption. We will assume later that such a dual Wets–Witzgall algorithm is available for use as a subroutine when we present the procedure for solving the neighborhood problem in Y. In the neighborhood problem procedure, a modified Wets–Witzgall algorithm is needed to find frames for pointed cones only. As we have noted, in such cases a frame is essentially equivalent to the set of extreme directions of $W(S)$. However, since Wets and

Witzgall did not make this observation, and since they described their algorithm as a means for finding a frame, to avoid confusion, we maintain consistency with their presentation by using the concept of a frame in our presentation. Furthermore, we shortly will also need the concept of a frame to describe some work of Dauer and Liu [9] and of Dauer [8] that is related to the work in this article.

Recall that we have assumed that we may find a point $y \in Y_{\mathrm{ex}}$ and a point $z \in Z_{\mathrm{ex}}$ such that $y = Dz$. Assuming that we have done so, let $\bar{R}$ denote the $p \times (n - m)$ reduced cost matrix corresponding to an $m \times m$ basis matrix B associated with z (cf. Section 3).

Let $R = \{r^j \mid j = 1, 2, \ldots, h\}$ denote the set of columns of $\bar{R}$, where $h = n - m$. Dauer and Liu [9, Theorem 3.1] have claimed that since $y \in Y_{\mathrm{ex}}$, the image under D of the edge E_k of Z defined by raising the kth nonbasic variable in z to a positive level, where $k \in \{1, 2, \ldots, h\}$, is contained in an edge of Y if and only if r^k is an element of a frame for $W(R)$. Notice that since $y \in Y_{\mathrm{ex}}$, the cone $W(R)$ is pointed. Therefore, from Remark 4.1, this claim can be restated as saying that since $y \in Y_{\mathrm{ex}}$, the set of points $D[E_k]$ is contained in an edge of Y if and only if, for any frame T for $W(R)$, there exists some $t \in T$ such that $r^k = \theta t$ for some $\theta > 0$. Later, Dauer [8, p. 287] claimed that when there exist two adjacent vertices z and $\bar{z}$ of Z that satisfy $Dz = D\bar{z} = y$, then 'the reduced cost matrices for both of these vertices need to be analyzed in order to obtain all of the potential edges of Y at y'. In the same paper, Dauer [8, p. 285] also stated that when the point $z \in Z_{\mathrm{ex}}$ chosen such that $Dz = y$ is degenerate, then 'one must determine which simplex tableaux' for z 'need to be analyzed in order to determine the edges of Y' at y. These latter two claims, although not stated more precisely in [8], seem to contradict the initial claim of Dauer and Liu [9, Theorem 3.1].

In what follows, we derive the theory that we will need to develop well-defined mechanics for identifying all of the edges of Y emanating from y and the extreme point neighbors to y in Y that lie at the endpoints of these edges, i.e., to solve the neighborhood problem in Y. By using these mechanics, we will be able to pivot from extreme point to extreme point in Y, and we will also resolve the apparent contradictions in [8] and [9]. Towards this end, we present two key results.

THEOREM 4.1. *Let $W \subseteq \Re^n$ be a nonempty convex set, and let $V = C[W] = \{Cw \mid w \in W\}$, where C is a $p \times n$ matrix. Assume that $v \in V$, that $d_v \in \Re^p$ is nonzero, and that for some $\gamma > 0$, $(v + \theta d_v) \in V$ for all θ satisfying $0 < \theta \leq \gamma$. Let w be any point in W such that $Cw = v$. Then there exists a nonzero vector $d_w \in \Re^n$ such that for all θ satisfying $0 < \theta \leq \gamma$,*

$$(a) \quad (w + \theta d_w) \in W$$

and

$$(b) \quad C(w + \theta d_w) = (v + \theta d_v)\,.$$

Proof. Choose $\bar{\theta} = \gamma$. Then, by assumption, $(v + \bar{\theta} d_v) \in V$. By definition of V, this implies that for some $\bar{w} \in W$, $C\bar{w} = (v + \bar{\theta} d_v) \in V$. Let $d_w = (1/\bar{\theta})(\bar{w} - w)$.

Notice that since $\bar{\theta} > 0$ and $d_v \neq 0$, $\bar{\theta} d_v \neq 0$. Since $v = Cw = C\bar{w} - \bar{\theta} d_v$, this implies that $\bar{w} \neq w$. Together with the fact that $\bar{\theta} > 0$, this implies that d_w is nonzero. Furthermore,

$$\begin{aligned}(w + \bar{\theta} d_w) &= w + \bar{\theta}(1/\bar{\theta})(\bar{w} - w) \\ &= \bar{w} \, .\end{aligned}$$

Since $\bar{w} \in W$, this implies that $(w + \bar{\theta} d_w) \in W$, i.e. $(w + \gamma d_w) \in W$. By the convexity of W, it follows that, for any scalar t such that $0 < t \leqslant 1$,

$$(1 - t)w + t(w + \gamma d_w) \in W \, .$$

This implies that for any θ satisfying $0 < \theta \leqslant \gamma$,

$$(w + \theta d_w) \in W \, .$$

Now suppose that θ satisfies $0 < \theta \leqslant \gamma$. Then

$$\begin{aligned}C(w + \theta d_w) &= Cw + \theta C d_w \\ &= v + \theta C[(1/\bar{\theta})(\bar{w} - w)] \\ &= v + (\theta/\bar{\theta})[C\bar{w} - Cw] \\ &= v + (\theta/\bar{\theta})[C\bar{w} - v] \\ &= v + (\theta/\bar{\theta})[(v + \bar{\theta} d_v) - v] \\ &= v + \theta d_v \, ,\end{aligned}$$

where the second-to-last equality follows from the fact that $C\bar{w} = (v + \bar{\theta} d_v)$. The proof is complete. □

Under more restrictive assumptions, the vector d_w in Theorem 4.1 can be chosen to have some additional properties beyond those given in the theorem. In particular, we have the following result.

THEOREM 4.2. *In addition to the assumptions in Theorem 4.1, assume that W is a nonempty, compact polyhedron, that v is an extreme point of V, and that for all θ satisfying $0 \leqslant \theta \leqslant \gamma$, $(v + \theta d_v)$ lies on a nontrivial edge E of V. In addition, let w be* any *extreme point of W such that $Cw = v$. Then, for any $\beta > 0$, a nonzero vector d_w may be chosen such that for all θ satisfying $0 < \theta \leqslant \beta$,*

(*a*) $(w + \theta d_w)$ *lies on an edge of* W ,

and

(*b*) $C(w + \theta d_w)$ *is distinct from* Cw *and lies on the edge* E *of* V .

Proof. First, notice that since W is nonempty and compact and v is an extreme point of V, by [4] there exists at least one extreme point $\bar{w}$ of W such that $C\bar{w} = v$. Let w represent *any* such extreme point.

Let $W^{-1}(E) = \{\bar{w} \in W \mid C\bar{w} \in E\}$. Since the dimension of E is one, from [4], $W^{-1}(E)$ is a face of W of dimension one or larger. Notice by the definition of an extreme point that since $w \in W^{-1}(E)$ and w is an extreme point of W, w is also an extreme point of $W^{-1}(E)$. Either (i) $W^{-1}(E)$ has dimension one or (ii) $W^{-1}(E)$ has dimension two or more.

Case (*i*): $W^{-1}(E)$ has dimension one. Then, since W is compact, $W^{-1}(E)$ is a nontrivial edge of W with distinct endpoints w and $\hat{w}$ for some $\hat{w}$ that is an extreme point of W. Let β denote an arbitrary positive number, and set

$$d_w = (1/\beta)(\hat{w} - w) . \tag{11}$$

Since $\beta > 0$ and $\hat{w} \neq w$, from (11) it follows that $d_w \neq 0$. Suppose that θ satisfies $0 < \theta \leqslant \beta$. Then $0 < \theta/\beta \leqslant 1$. Therefore, the point

$$(1 - \theta/\beta)w + (\theta/\beta)\hat{w}$$

is a convex combination of w and $\hat{w}$, so that it lies on $W^{-1}(E)$. Notice that

$$\begin{aligned}(1 - \theta/\beta)w + (\theta/\beta)\hat{w} &= w + (\theta/\beta)(\hat{w} - w) \\ &= w + \theta d_w ,\end{aligned}$$

where the second equation follows from (11). Since $W^{-1}(E)$ is an edge of W, the latter two statements imply that $(w + \theta d_w)$ lies on an edge of W.

Since $Cw = v$, it follows that

$$C(w + \theta d_w) = v + \theta C d_w . \tag{12}$$

Furthermore,

$$\begin{aligned}v + \theta C d_w &= v + (\theta/\beta)(C\hat{w} - Cw) \\ &= v + (\theta/\beta)(C\hat{w} - v) \\ &= [1 - (\theta/\beta)]v + (\theta/\beta)C\hat{w} ,\end{aligned} \tag{13}$$

where the first equation follows from (11), and the second equation follows from the fact that $Cw = v$. Notice from (12) and (13) that $C(w + \theta d_w)$ is a convex combination of v and $C\hat{w}$, both of which belong to E. Therefore, $C(w + \theta d_w)$ also belongs to E.

Since $C[W^{-1}(E)] = E$, and E is nontrivial, there must exist some $\bar{\theta}$ such that $0 < \bar{\theta} \leqslant \beta$ and $v = Cw \neq C(w + \bar{\theta} d_w)$. Therefore, $Cd_w \neq 0$. Since $\theta > 0$, this implies that $C(w + \theta d_w)$ is distinct from Cw.

Case (*ii*): $W^{-1}(E)$ has dimension two or more. Let $\{w^1, w^2, \ldots, w^q\}$ denote the extreme points of $W^{-1}(E)$ that are adjacent to w in $W^{-1}(E)$. Then, since $W^{-1}(E)$ is compact and has dimension two or more, $q \geqslant 1$.

We claim that for at least one $i \in \{1, 2, \ldots, q\}$, $Cw \neq Cw^i$. The following argument shows this claim. Since $(v + \theta d_v) \in E$ for all θ such that $0 \leqslant \theta \leqslant \gamma$, where

$\gamma > 0$ and $d_v \neq 0$, we may choose a value $\hat{\theta}$ for θ such that $0 < \hat{\theta} \leqslant \gamma$ and $\hat{v} \overset{\triangle}{=} (v + \hat{\theta} d_v) \in E$, where $\hat{v} \neq v$. Because $\hat{v} \in E$, we may choose a point $\hat{w} \in W^{-1}(E)$ such that $C\hat{w} = \hat{v}$. Let

$$d = (1/\hat{\theta})(\hat{w} - w) . \tag{14}$$

We see that since $\hat{\theta} > 0$ and $\hat{v} \neq v$, $d \neq 0$. Furthermore, by (14),

$$w + \hat{\theta} d = \hat{w} ,$$

so that

$$\begin{aligned} C\hat{w} &= Cw + \hat{\theta} Cd \\ &= \hat{v} \\ &= v + \hat{\theta} Cd , \end{aligned}$$

where the second equality follows by the choice of $\hat{w}$, and the third equality follows from the assumption that $Cw = v$. Since $\hat{v} \neq v$, the latter equation implies that $Cd \neq 0$.

For any θ such that $0 < \theta < \hat{\theta}$, from (14), $(w + \theta d)$ is a convex combination of w, $\hat{w} \in W^{-1}(E)$. Therefore, d represents a nonzero feasible direction of movement in $W^{-1}(E)$ at the extreme point w of $W^{-1}(E)$. It is easy to show that this implies that for some $\lambda_i \geqslant 0$, $i = 1, 2, \ldots, q$, not all of which are zero,

$$d = \sum_{i=1}^{q} \lambda_i (w^i - w) .$$

Therefore,

$$Cd = \sum_{i=1}^{q} \lambda_i (Cw^i - Cw) .$$

Since $Cd \neq 0$, $Cw^i \neq Cw$ for at least one $i \in \{1, 2, \ldots, q\}$, so that the claim is established.

By the claim, we may assume without loss of generality that $Cw^1 \neq Cw$. Then w and w^1 are distinct endpoints of a nontrivial edge of $W^{-1}(E) \subseteq W$ such that $Cw \neq Cw^1$. By using these properties of w and w^1 and arguments taken from Case (i) with w^1 playing the role of $\hat{w}$, we easily obtain the remainder of the proof. □

Suppose that $y \in Y_{\text{ex}}$ and that the goal is to solve the neighborhood problem at y. By applying Theorems 4.1 and 4.2 with Z, D, Y and y playing the roles of W, C, V and v, respectively, we see that to find any nontrivial edge E of Y emanating from y, we may choose *any* extreme point z of Z such that $Dz = y$ and search among the edges of Z emanating from z for an edge that contains a point that is mapped by D into the relative interior of E. Notice that this statement holds *even if* the chosen extreme point z of Z is degenerate and *even if* there exist multiple extreme points of Z that are mapped by D into y. Furthermore, given the edge E, Theorems 4.1 and 4.2 guarantee that for the chosen z, there *will exist* an edge of Z emanating from z

that contains a point that is mapped by D into the relative interior of E. Thus, Theorems 4.1 and 4.2 resolve the apparent contradictions and confusions in [8] and [9]. They also establish a framework for developing mechanics for solving the neighborhood problem in Y, as we shall now see. In this way, pivoting among the extreme points of Y in a simplex-like manner will become possible.

Assume as before that we have found a point $y \in Y_{\text{ex}}$ and a point $z \in Z_{\text{ex}}$ such that $y = Dz$. As before, let $\bar{R}$ denote the $p \times (n-m)$ reduced cost matrix corresponding to an $m \times m$ basis matrix B associated with z. In addition, as in Section 3, let N denote the $(n-m) \times m$ matrix consisting of the columns of A that do not belong to B, let $z_B \in \Re^m$ denote the vector of basic variables corresponding to B, let $z_N = 0 \in \Re^{n-m}$ denote the vector of nonbasic variables, and let D_B denote the $p \times m$ matrix that consists of the columns of D corresponding to B.

These data can be conveniently summarized in a $(p+m) \times (n+1)$ *decision-outcome set simplex tableau* T given by

$$T = \left[\begin{array}{ccc:c} 0 & \bar{R} & & D_B B^{-1} b \\ \cdots & \cdots & \cdots & \cdots \\ I_m & B^{-1}N & & B^{-1}b \end{array}\right], \tag{15}$$

where I_m denotes the $m \times m$ identity matrix, the first m columns of T correspond to z_B, the next $(n-m)$ columns of T correspond to $z_N = 0$, and the rightmost column of T stores the current values of y and z_B. With T as an aid, we can now state the procedure for solving the neighborhood problem for Y at y.

Procedure for neighborhood problem solution in Y

Step 1. Given $y \in Y_{\text{ex}}$, find any point $z \in Z_{\text{ex}}$ such that $Dz = y$. Construct a decision-outcome set simplex tableau T (cf. (15)) corresponding to z.

Step 2. Using T and any necessary alternate forms of T, find all nonbasic columns corresponding to z such that the execution of a simplex method pivot in each of these columns leads to an extreme point neighbor to z in Z that is distinct from z. Let $R = \{r^1, r^2, \ldots, r^t\}$ represent the set of all p-vectors obtained by deleting all entries of each of these nonbasic columns beyond the pth entry.

Step 3. Construct a frame F for $W(R)$. Let $F = \{r^k \mid k \in I\}$ denote this frame, where $I \subseteq \{1, 2, \ldots, t\}$.

Step 4. For each $k \in I$, find an optimal solution $(\bar{z}^k, \bar{\theta}_k)$ to the linear program (LP_k) given by

$$\max \bar{\theta}\,,$$

s.t.

$$D\bar{z} - \bar{\theta} r^k = y\,,$$

$$A\bar{z} = b\ ,$$
$$\bar{z}, \bar{\theta} \geqslant 0\ .$$

For each $k \in I$, set

$$E_k = \{y + \bar{\theta} r^k \mid 0 \leqslant \bar{\theta} \leqslant \bar{\theta}_k\}$$

and

$$y^k = y + \bar{\theta}_k r^k\ .$$

Then the set of all nontrivial edges of Y emanating from y is given by $\{E_k \mid k \in I\}$. Furthermore, $\{y^k \mid k \in I\}$ is the set of all distinct extreme point neighbors in Y to y, where, for each $k \in I$, $y^k \in E_k$.

Given $y \in Y_{\text{ex}}$, to find a point $z \in Z_{\text{ex}}$ such that $Dz = y$ as called for by Step 1 of the procedure, one can, for instance, solve the linear program (LP_y) given by

$$\min \sum_{i=1}^{p} \bar{u}_i\ ,$$
s.t.
$$D\bar{z} - \bar{u} = y\ ,$$
$$A\bar{z} = b\ ,$$
$$\bar{z}, \bar{u} \geqslant 0$$

for an optimal basic solution $(\bar{z}^*, \bar{u}^*) = (\bar{z}^*, 0) \in \Re^{n+p}$. Then, with $z = \bar{z}^*$, $Dz = y$ holds, and it is easy to verify that $z \in Z_{\text{ex}}$ also holds.

To execute Step 2, one must first clarify the basic feasible solution displayed in tableau T for z as nondegenerate or degenerate. If $B^{-1}b > 0$, this basic feasible solution is nondegenerate. In this case, a simplex method pivot can be executed in each of the nonbasic column of T to yield an extreme point neighbor in Z to z, so that R will contain all of the $(n - m)$ columns of $\bar{R}$. If $B^{-1}b > 0$ does not hold, then the basic feasible solution displayed in tableau T for z is degenerate. In this case, z may have fewer or more than $(n - m)$ distinct neighboring extreme points in Z. To find all nonbasic columns corresponding to z such that for each column, for some tableau representing z, the execution of a simplex method pivot in the column leads to an extreme point neighbor to z in Z, one can use, for instance, the ‘transition node pivoting’ procedure of Gal and Geue [13] (also see Geue [14]). In this procedure, certain alternate tableaus T corresponding to z are generated, and the columns of R in Step 2 are gathered from certain columns of these tableaus. For details, see [13, 14].

The frame F called for in Step 3 of the procedure can be generated, in the absence of certain ‘degeneracies’, by the primal algorithm of Wets and Witzall [28]. As mentioned earlier, one can also use an appropriate ‘dual’ algorithm. We have developed a dual algorithm for finding F in Step 3 that requires no special nondegeneracy assumption. From Dauer and Liu [9], the edges of Y emanating from

y have directions that are in one-to-one correspondence with the elements of the frame F constructed in Step 3.

Step 4 is taken directly from Dauer and Liu [9]. Notice in Step 4 that if $i \in I$ and one simply desires to move (or 'pivot') from y along the edge E_i until the neighboring extreme point y^i to y in Y is found, one can solve a single linear program (LP_i) for an optimal solution $(\bar{z}^i, \bar{\theta}_i)$. The neighboring extreme point y^i in Y to y is then given by

$$y^i = y + \bar{\theta}_i r^i .$$

5. Example

To illustrate the procedure for solving the neighborhood problem in Y (and, as a result, for pivoting in Y), consider Example 4.1 from [9]. In this example, $m = 8$, $n = 11$ and $p = 2$. The problem data are given by

$$A = \begin{bmatrix} 9 & 9 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 8 & 1 & 8 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 8 & 8 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 7 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 7 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 1 & 7 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 81 \\ 72 \\ 72 \\ 9 \\ 9 \\ 9 \\ 8 \\ 8 \end{bmatrix}$$

and

$$D = \begin{bmatrix} 1 & 0 & 1/9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1/9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} .$$

In this case, $Z \subseteq \Re^{11}$ and has eleven extreme points, and $Y \subseteq \Re^2$ has five extreme points.

Suppose in Step 1 of the procedure that we are given the extreme point

$$y^T = \left(\frac{1}{9},\ 8\,\frac{1}{9}\right) \tag{16}$$

of Y. For instance, y may have been generated by the initialization procedure given in this article or by a previous outcome space pivot to y from some neighbor of y in Y. The steps of the neighborhood problem procedure with y given by (16) proceed as follows.

Step 1. Solving linear program (LP_y), we obtain an optimal basic solution $((\bar{z}^*)^T, 0, 0)$, where

$$(\bar{z}^*)^T = (0, 8, 1,\ 7, 56, 0, 0, 48, 6, 8, 0) .$$

Then, with $z = \bar{z}^*$, $z \in Z_{\text{ex}}$ and $Dz = y$. The extreme point z of Z is degenerate. One basis matrix B for z uses basic variables $z_4, z_3, z_8, z_6, z_2, z_9, z_{10}$ and z_5. The decision-outcome set simplex tableau T for z with this basis matrix B is given by

$$T = \left[\begin{array}{cccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2/9 & 1/9 & 1/9 & \vdots & 1/9 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -7/9 & 1/9 & -8/9 & \vdots & 73/9 \\
. & . & . & . & . & . & . & . & .... & ... & .... & . & .... \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5 & 2 & -7 & \vdots & 7 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 7 & -1 & -1 & \vdots & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 6 & -1 & 6 & \vdots & 48 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -55 & \textcircled{8} & 0 & \vdots & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & \vdots & 8 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 48 & -7 & -6 & \vdots & 6 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & \vdots & 8 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -48 & 8 & 7 & \vdots & 56
\end{array}\right],$$

where the meaning of the circled number will be explained below, and where the three nonbasic columns in the far right side of the body of T correspond to the variables z_1, z_7 and z_{11}, respectively.

Step 2. Since tableau T displays a degenerate basic feasible solution, we use the transition node pivoting procedure of Gal and Geue [13] to execute this step. This procedure tells us that by performing a simplex method pivot in T in either of the nonbasic columns corresponding to z_1 or to z_{11}, we could generate a distinct neighboring extreme point in Z to z. Thus,

$$(r^1)^T = (2/9, -7/9)$$

and

$$(r^2)^T = (1/9, -8/9)$$

each belong to R. Furthermore, the Gal and Geue procedure calls for generating the alternate decision-outcome set simplex tableau T' for z obtained by performing a simplex method pivot using the circled number 8 shown in T as the pivot element. Upon so doing, a third element r^3, given by

$$(r^3)^T = (71/72, -1/72)\,,$$

of R is generated, and the Gal and Geue procedure stops.

Step 3. Using the 'dual' algorithm mentioned earlier that we have devised, we find that a frame F for $W(R)$ is given by

$$F = \{r^2, r^3\}\,.$$

Thus, we set $I = \{2, 3\}$.

Step 4. With $k = 2$ and with $k = 3$, linear program (LP_k) has optimal solutions

$$[(\bar{z}^2)^T, \bar{\theta}_2] = (0, 0, 9, 63, 0, 0, 0, 0, 54, 8, 8, 8)$$

and

$$[(\bar{z}^3)^T, \bar{\theta}_3] = (4/5, 8, 9/10, 0, 252/5, 0, 11/2, 487/10, 61/10, 36/5, 0, 4/5)\,,$$

respectively. Therefore, y has two nontrivial edges emanating from it in Y. These edges are given by

$$E_1 = \{(1/9, 73/9) + \bar{\theta}(1/9, -8/9) \mid 0 \leqslant \bar{\theta} \leqslant 8\}$$

and

$$E_2 = \{(1/9, 73/9) + \bar{\theta}(71/72, -1/72) \mid 0 \leqslant \bar{\theta} \leqslant 4/5\},$$

and $(y^1)^T = (1, 1)$ and $(y^2)^T = (9/10, 81/10)$ are the two extreme point neighbors to y in Y, where $y^k \in E_k$, $k = 1, 2$.

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